A New Perspective on Quality Evaluation for Control Systems with Stochastic Timing

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FACULTY OF ENGINEERING

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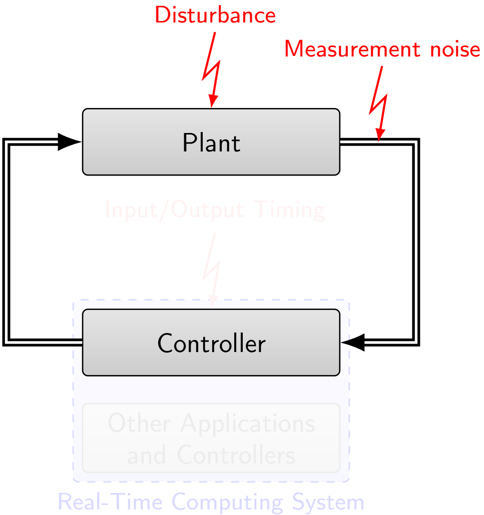


Quality of Control:

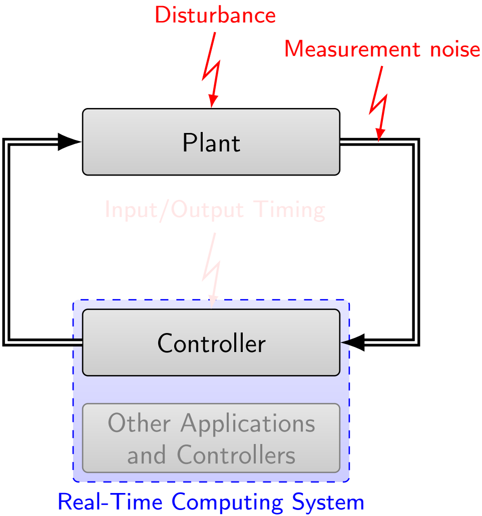
How well does the control system work under

- random disturbance
- **I/O timing**?

Time-varying situation:

- execution conditions
- disturbance amplitude
- reference trajectory

$\rightsquigarrow$ **Quality is time-varying**



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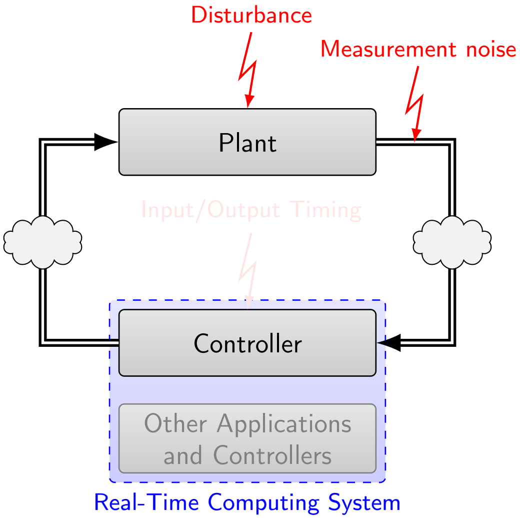

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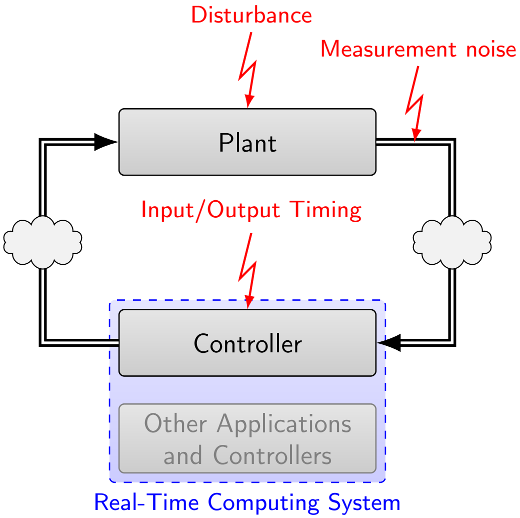

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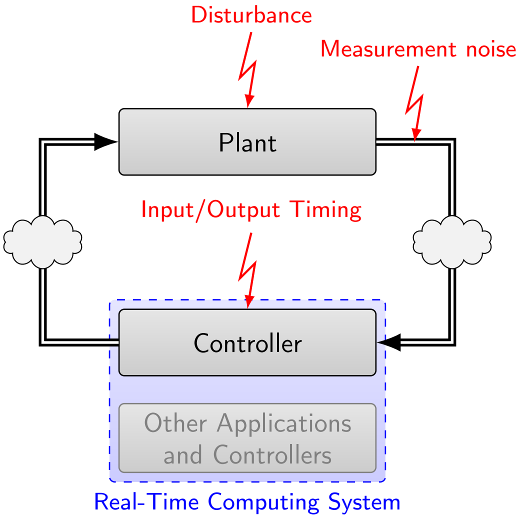

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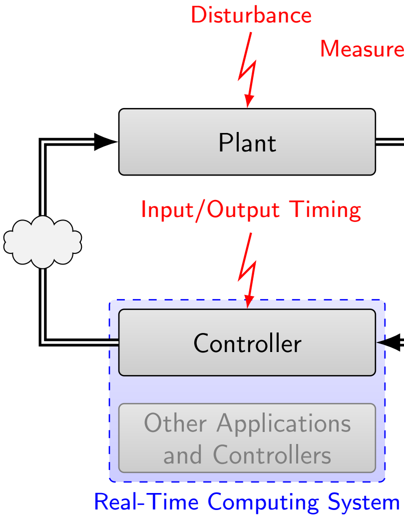

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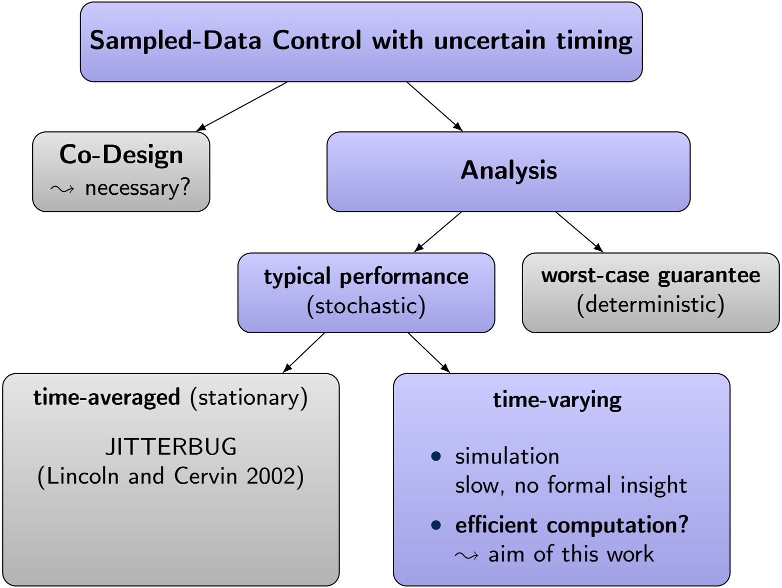
Sampled-Data Control with uncertain timing
Co-Design
⤳ necessary?
Analysis
typical performance
(stochastic)
worst-case guarantee
(deterministic)
time-averaged (stationary)
JITTERBUG
(Lincoln and Cervin 2002)
time-varying
• simulation
slow, no formal insight
• efficient computation?
⤳ aim of this work

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- continuous-time MIMO plant (linear, time-invariant)

$$\dot{x}_\mathrm{p}(t) = A_\mathrm{p} x_\mathrm{p}(t) + B_\mathrm{p} u(t) + G_\mathrm{p} d(t), \quad x_\mathrm{p}(0) = 0,$$
$$y(t) = C_\mathrm{p} x_\mathrm{p}(t) + w_\mathrm{p}(t)$$

- $d(t)$: stochastic disturbance (white noise, time-varying covariance $H(t)$)
- $w_\mathrm{p}(t)$: measurement noise

- discrete-time controller (linear + reference trajectory), sampling time T

$$x_\mathrm{d}[k+1] = A_\mathrm{d}[k] x_\mathrm{d}[k] + B_\mathrm{d}[k] y[k] + f_\mathrm{d}[k],$$
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  - random, independent of disturbance and measurement noise

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- quality via quadratic cost function: deviation from reference $x_\mathrm{r}, u_\mathrm{r}$

$$\begin{aligned} J(t) =& (x_\mathrm{p}(t) - x_\mathrm{r}(t))^T \tilde{Q} (x_\mathrm{p}(t) - x_\mathrm{r}(t)) \\ &+ (u(t) - u_\mathrm{r}(t))^T \tilde{R} (u(t) - u_\mathrm{r}(t)) \end{aligned}$$

  with $x_r(t), u_r(t)$ known a priori.

- **desired result:** expected cost $\mathcal{E}\{J(t)\}$ (time-varying)

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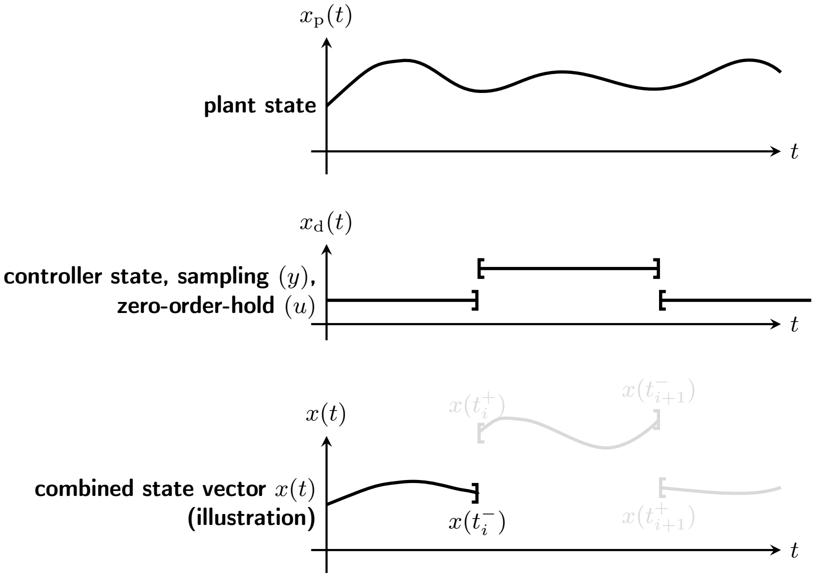
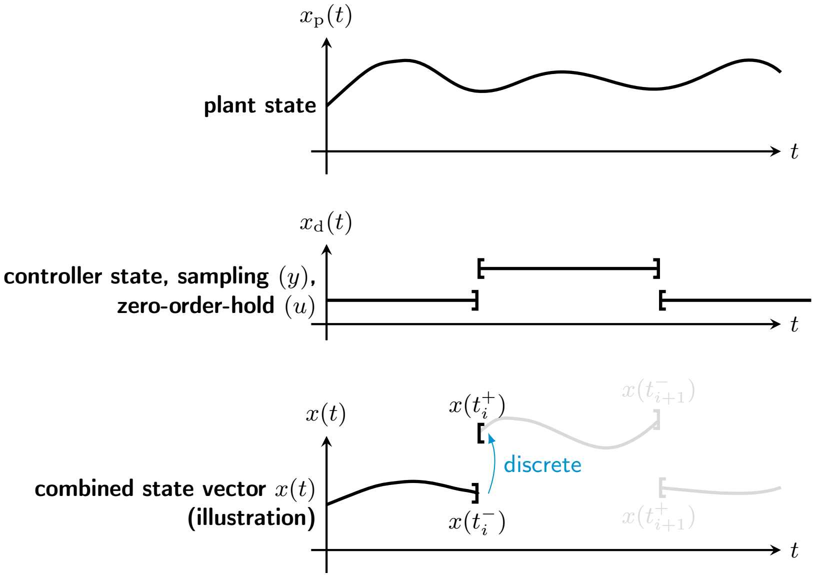

$x_\mathrm{p}(t)$
plant state
t
$x_\mathrm{d}(t)$
controller state, sampling (y),
zero-order-hold (u)
t
$x(t)$
$x(t_i^+)$
$x(t_{i+1}^-)$
combined state vector $x(t)$
(illustration)
$x(t_i^-)$
$x(t_{i+1}^+)$
t


$x_p(t)$
plant state
t
$x_d(t)$
controller state, sampling (y),
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t
$x(t)$
$x(t_i^+)$
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discrete
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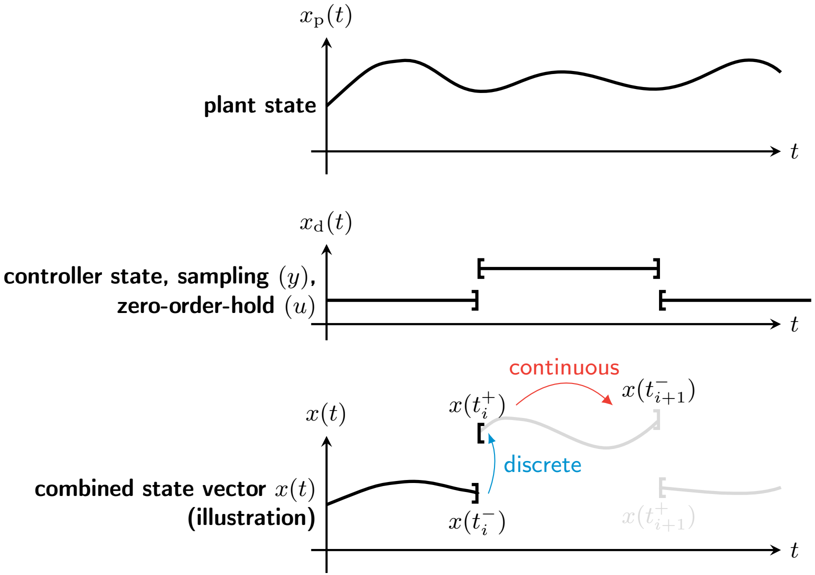
$x(t_i^-)$
$x(t_{i+1}^+)$
t



$x_\mathrm{p}(t)$
plant state
t
$x_\mathrm{d}(t)$
controller state, sampling (y),
zero-order-hold (u)
t
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$x(t_i^+)$
$x(t_{i+1}^-)$
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combined state vector $x(t)$
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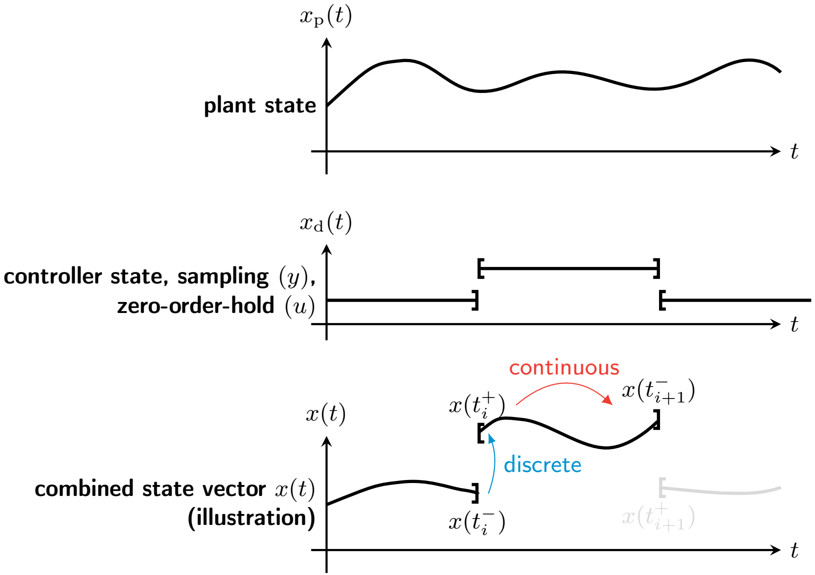
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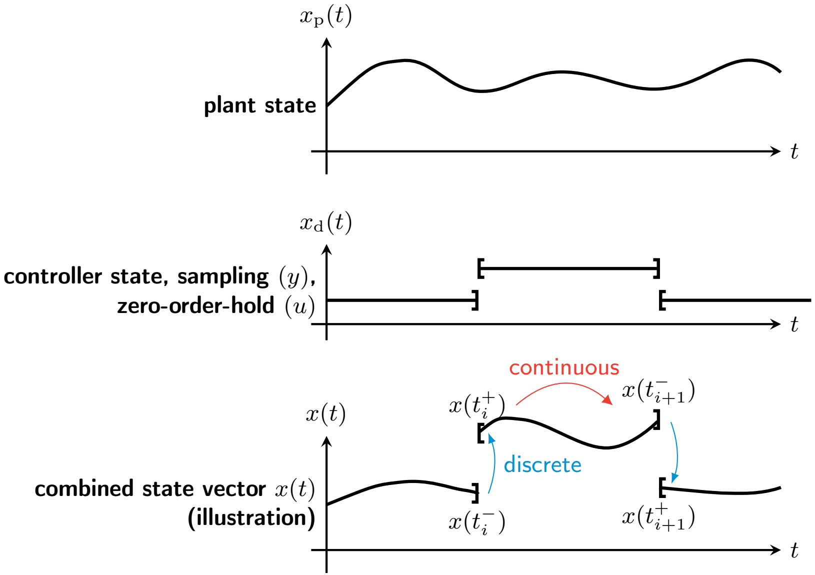
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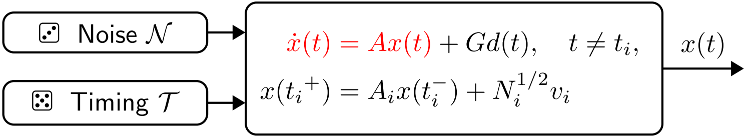
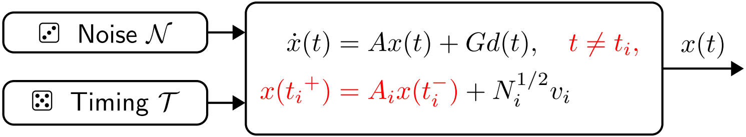


- continuous dynamics
- interrupted by discrete jumps (sample, update controller, actuate)

- input: noise $\mathcal{N}$
  - $d(t)$: disturbance
  - v_i: measurement randomness (covariance I)
- formalized IO timing $\mathcal{T}$: $\Delta t_{\ldots} \mapsto \underbrace{(t_i, A_i, N_i)_{i \in \mathbb{N}}}_{\mathcal{T}}$



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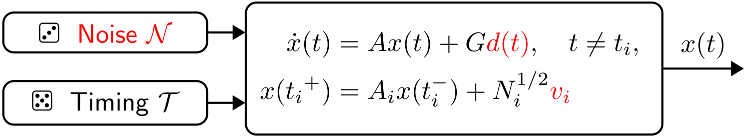

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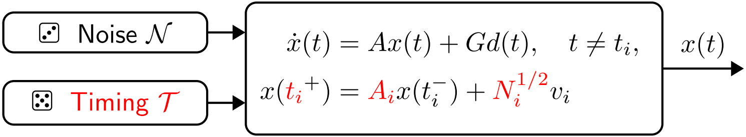

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- **assumption**: I/O timing $\mathcal{T}$ known a priori.
- $\rightsquigarrow$ remaining randomness: noise $\mathcal{N}$
- **result**: expected cost $\mathcal{E}\{J(t)\}$ w.r.t. $\mathcal{N}$

Consider "covariance"matrix $P(t) := \mathcal{E}\left\{x(t)x(t)^T\right\}$ of combined state.

(No real covariance: $\mathcal{E}\{x(t)\} \neq 0$)

temporal evolution:

1. discrete from $x(t_i^-)$ to $x(t_i^+)$

$$P(t_i^+) = A_i P(t_i^-) A_i^T + N_i$$

2. continuous from $x(t_i^+)$ to $x(t_{i+1}^-)$

$$P(t_{i+1}^-) = \mathrm{e}^{A\Delta} P(t_i^+) \, \mathrm{e}^{A^T\Delta} + \int_0^\Delta \mathrm{e}^{A\tau} G H G^T (\mathrm{e}^{A\tau})^T \, \mathrm{d}\tau$$

with $\Delta = t_{i+1}^- - t_i^+$.

- iterate (1) + (2) $\rightsquigarrow$ whole time axis
- $\approx$ **stochastic time discretization at *actual* I/O times**

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- rewrite cost function as

$$J(t) = x^T(t)\, Q(t)\, x(t)$$

- $Q(t)$ time-varying, but deterministic (contains reference trajectory)

- compute cost from covariance matrix:

$$\mathcal{E}\{J(t)\} = \mathcal{E}\left\{x^T(t)\, Q(t)\, x(t)\right\} = \operatorname{trace}\Big(Q(t)\, \underbrace{\mathcal{E}\left\{x(t)x^T(t)\right\}}_{P(t)}\Big)$$

- rewrite cost function as

$$J(t) = x^T(t)\,Q(t)\,x(t)$$

- $Q(t)$ time-varying, but deterministic (contains reference trajectory)
- compute cost from covariance matrix:

$$\mathcal{E}\{J(t)\} = \mathcal{E}\left\{x^T(t)\,Q(t)\,x(t)\right\} = \mathrm{trace}\Big(Q(t)\;\underbrace{\mathcal{E}\left\{x(t)x^T(t)\right\}}_{P(t)}\Big)$$

Result:

- dynamics of covariance $P(t)$
- $\rightsquigarrow$ time-varying Quality of Control via cost $\mathcal{E}\{J(t)\}$

Classical Simulation (for comparison):

- pseudo-randomness
- ensemble mean over many runs $\rightarrow$ inefficient
- only stochastic convergence (in probability)

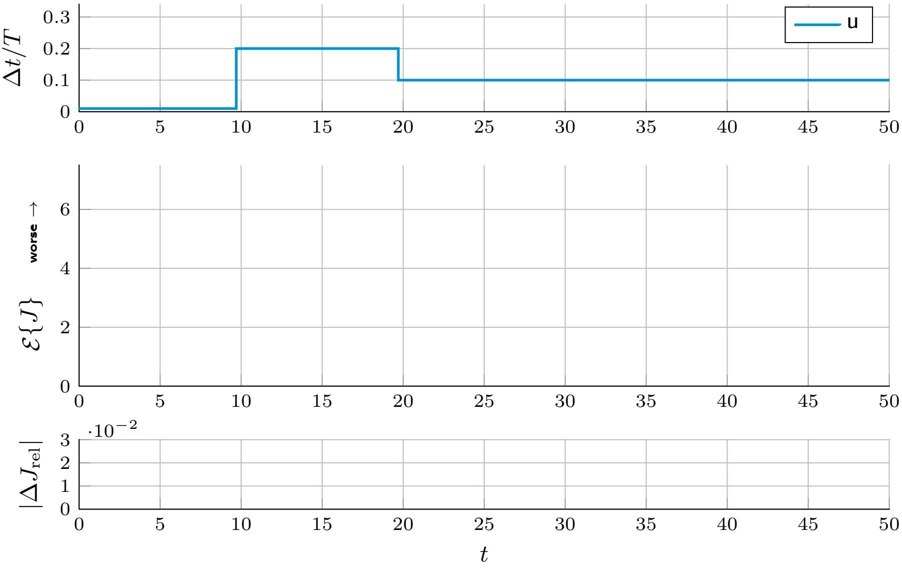
Example:

- linearized inverse pendulum (SISO)
- discrete-time observer-based state feedback, $T = 0.2$
- pole placement, dangerously fast choice: $|\lambda_{\mathrm{control,continuous}}| \approx 5|\lambda_{\mathrm{plant}}|$
- deterministic delay of u and y

Result:

- model run-time below 1 second
- simulation run-time over 5 hours for 3 % relative error

(both implementations may be further optimized)


u
$\Delta t/T$
0.3
0.2
0.1
0
0 5 10 15 20 25 30 35 40 45 50
worse →
$\mathcal{E}\{J\}$
6
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0 5 10 15 20 25 30 35 40 45 50
$\cdot 10^{-2}$
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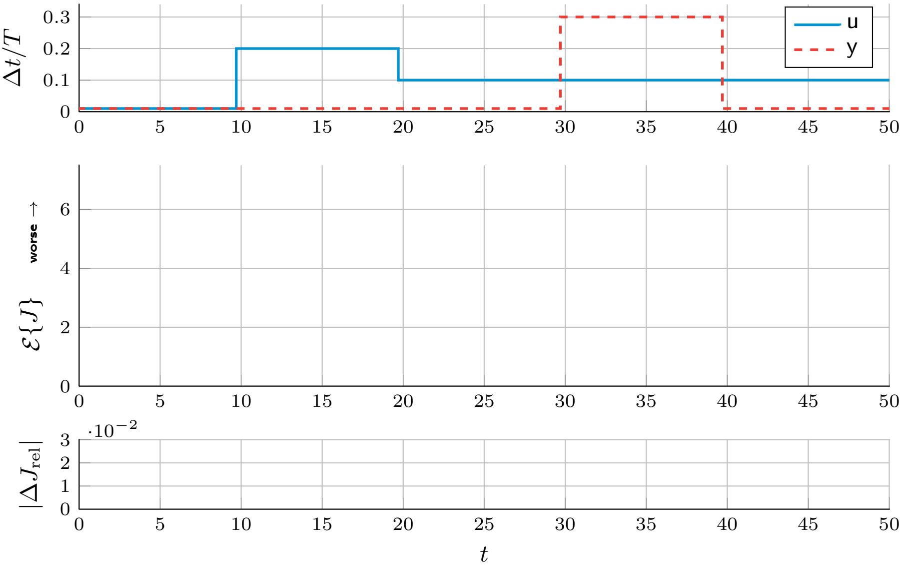
u
y
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worse →
$\mathcal{E}\{J\}$
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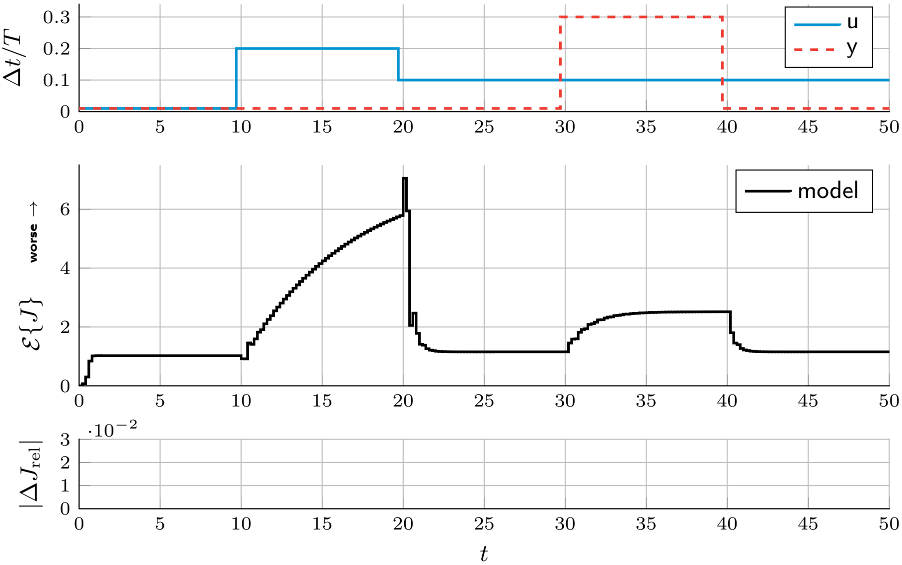

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$\Delta t/T$
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y
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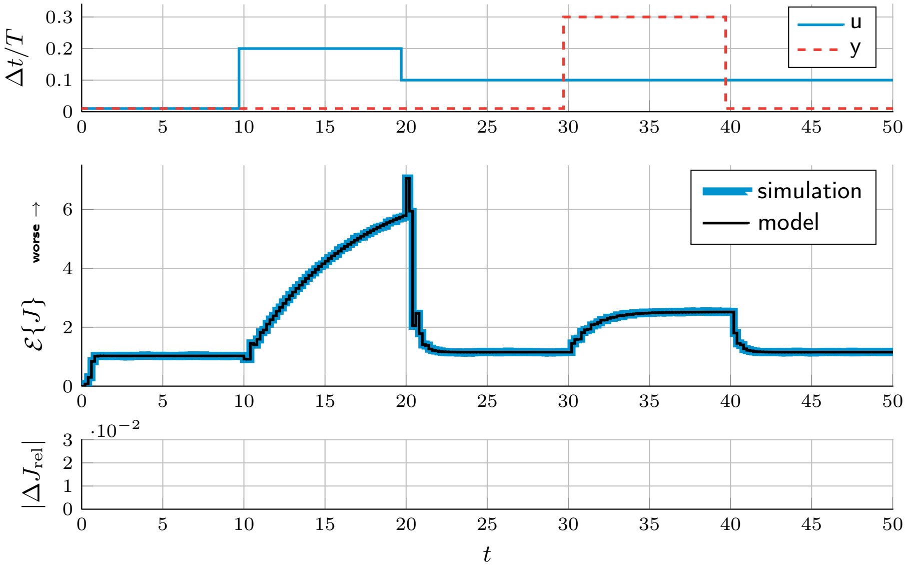
worse →
$\mathcal{E}\{J\}$
0 2 4 6
$\cdot 10^{-2}$
$|\Delta J_{\mathrm{rel}}|$
0 1 2 3
t



$\Delta t/T$
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simulation
model
$|\Delta J_{\mathrm{rel}}|$
$\cdot 10^{-2}$
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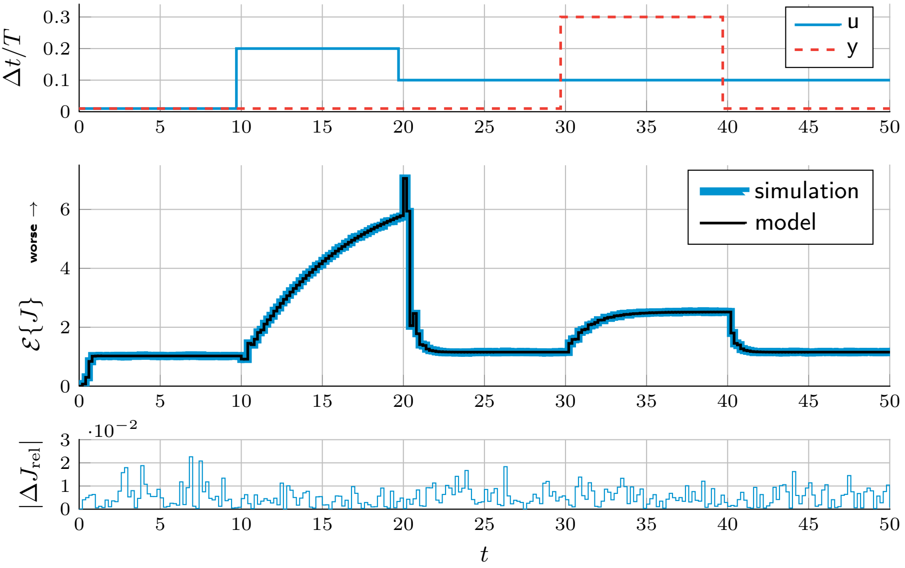
t



$\Delta t/T$
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worse →
$\mathcal{E}\{J\}$
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model
0 5 10 15 20 25 30 35 40 45 50
$|\Delta J_{\mathrm{rel}}|$
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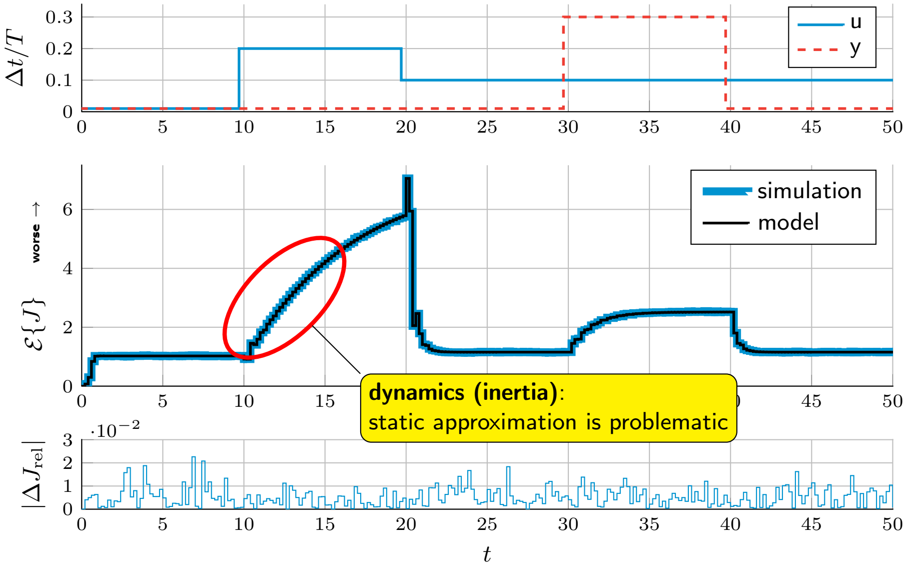

$\Delta t/T$
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y
worse →
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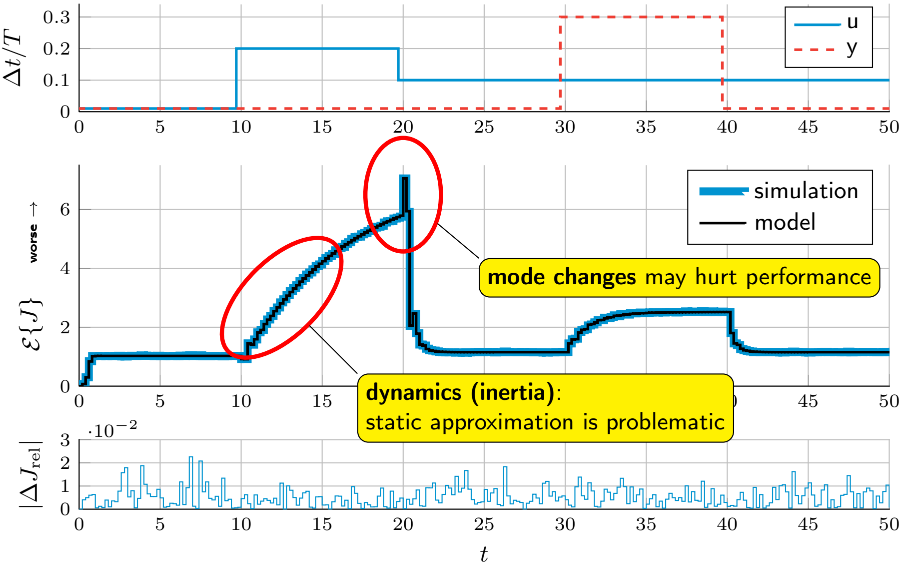
simulation
model
dynamics (inertia):
static approximation is problematic
$|\Delta J_{\mathrm{rel}}|$
$\cdot 10^{-2}$
3
2
1
0
t



$\Delta t/T$
u
y
$\mathcal{E}\{J\}$
worse →
simulation
model
mode changes may hurt performance
dynamics (inertia):
static approximation is problematic
$|\Delta J_{\text{rel}}|$
$\cdot 10^{-2}$
t

- current result F: covariance evolution from $t = a^+$ to $t = b^+$

$$P(b^+) = F_{(a,\,b]}(P(a^+), \mathcal{T}) \,, \quad P(0) = x_0 x_0^T$$

  for **deterministic** timing $\mathcal{T}$.

- Notation: $\underset{\text{random variables}}{\mathcal{E}} \{\ldots\}$
- **stochastic** timing:

$$P(t^+) = \underset{\mathcal{N},\mathcal{T}}{\mathcal{E}} \{x(t^+) x^T(t^+)\} = \cdots = \underset{\mathcal{T}}{\mathcal{E}} \left\{ F_{(0,\,t]} \left(x_0 x_0^T, \mathcal{T}\right) \right\}$$

  - noise $\mathcal{N}$ "eliminated"
  - weighted average over all timing sequences $\mathcal{T}$
  - complexity $\sim t \cdot \exp(t)$ ↯
- cost evaluation still from covariance

- current result F: covariance evolution from $t = a^+$ to $t = b^+$

$$P(b^+) = F_{(a,\,b]}(P(a^+), \mathcal{T}) \,, \quad P(0) = x_0 x_0^T$$

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for **deterministic** timing $\mathcal{T}$.

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$$P(t^+) = \underset{\mathcal{N},\mathcal{T}}{\mathcal{E}} \{x(t^+)x^T(t^+)\} = \dots = \underset{\mathcal{T}}{\mathcal{E}} \left\{ F_{(0,\,t]} \left(x_0 x_0^T, \mathcal{T}\right) \right\}$$

  - noise $\mathcal{N}$ "eliminated"
  - weighted average over all timing sequences $\mathcal{T}$
  - complexity $\sim t \cdot \exp(t)$ ↯

- cost evaluation still from covariance

Simplification for Stochastically Independent Time Segments:

- motivation: timing of control periods often (almost) independent
- timing with independent segments $t \in (\gamma_k, \gamma_{k+1}]$
- result:

$$P(\gamma_{k+1}^{+}) = \underset{\mathcal{T}_{(\gamma_k, \gamma_{k+1}]}}{\mathcal{E}} \left\{ \underset{(\gamma_k, \gamma_{k+1}]}{F} \left(P(\gamma_k^{+}), \mathcal{T}_{(\gamma_k, \gamma_{k+1}]} \right) \right\}$$

  - expectation only w.r.t. subsequence $\mathcal{T}_{(\gamma_k, \gamma_{k+1}]}$
  - complexity $\sim t \cdot \exp(\gamma_{k+1} - \gamma_k)$

Problem:

- real-time MIMO control loop
- effect of varying situations (timing) on Quality of Control?

Approach:

1. Linear Impulsive System (LIS)
2. covariance matrix dynamics ≈ stochastic discretization
3. generalization to stochastic timing

Theoretical Results:

- general LIS model for linear MIMO control
- *time-varying*, stochastic evaluation of Quality of Control

⇝ typically faster than random simulation

Practical Result:

- visualize effects of **timing changes** on **typical** performance

Future improvements:

- Markov chain for timing
- speed up computations: lookup table, (over-)approximation
  $\rightsquigarrow$ use for online timing adaptation

Related Challenges:

- **nonlinear** case
- **worst-case** analysis (with our LIS model?)

Thank you!
Any questions?

source code (GPL3) on our project website:
`http://qronos.de`

Appendix

Timing is no longer constant:

- increasing system complexity
  - many cheap asynchronous sensors
  - many applications
- strict timing is expensive (over-provisioning)
- deliberate trade-off: fixed timing vs. efficiency
  - adaptive scheduling
  - mode changes (mixed-criticality)

Timing is not quality:

- real-time systems designed for timing = "Quality of Service" (QoS)
- **actual goal is Quality of *Control* (QoC)!**
- $\rightsquigarrow$ relation of QoS and QoC?

Combined state of plant, controller and sampling:

$$x(t) := \left[x_{\mathrm{p}}^{T}(t) \;\; x_{\mathrm{d}}^{T}(t) \;\; y_{\mathrm{d}}^{T}(t) \;\; u^{T}(t) \;\; 1\right]^{T}, \quad x(0) = \left[0 \; \ldots \; 0 \;\; 1\right]^{T}$$

- $x_{\mathrm{p}}(t)$: plant
- $x_{\mathrm{d}}(t)$: controller, e.g., observer state $\hat{x}[k]$
- $y_{\mathrm{d}}(t)$: sampled measurement
- $u(t)$: control signal (zero-order-hold)

(the last three: piecewise constant)

- 1: auxiliary state for deterministic parts ($\mathcal{E}\{\cdot\} \neq 0$)

- continuous dynamics for $t \neq t_i$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\begin{bmatrix} x_\mathrm{p}(t) \\ x_\mathrm{d}(t) \\ y_\mathrm{d}(t) \\ u(t) \\ 1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} A_\mathrm{p} & 0 & 0 & B_\mathrm{p} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_\mathrm{p}(t) \\ x_\mathrm{d}(t) \\ y_\mathrm{d}(t) \\ u(t) \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} G_\mathrm{p} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{G} d(t)$$

❶ sample $y_j[k]$ at $t_i = kT + \Delta t_{\mathrm{y},j}[k]$

$$y_{\mathrm{d},j}(t_i^+) = e_j^T y(t_i^-) = e_j^T \left(C_\mathrm{p} x_\mathrm{p}(t_i^-) + \underbrace{w_\mathrm{p}(t_i^-)}_{\text{measurement noise}}\right)$$

❷ update controller state $x_\mathrm{d}[k+1]$ at $t_i = (k + \frac{1}{2})T$

$$x_\mathrm{d}(t_i^+) = x_\mathrm{d}[k+1] = A_\mathrm{d}[k] x_\mathrm{d}(t_i^-) + B_\mathrm{d}[k] y_\mathrm{d}(t_i^-) + \underbrace{f_\mathrm{d}[k] \cdot 1}_{\text{for reference traj.}}$$

$$\rightsquigarrow A_i = \dots, \quad N_i = 0$$

❸ output $u_j[k+1]$ at $t_i = (k+1)T + \Delta t_{\mathrm{u},j}[k]$

$$u_{\mathrm{d},j}(t_i^+) = u_j[k+1] = e_j^T (C_\mathrm{d}[k] x_\mathrm{d}(t_i^-) + \underbrace{g_\mathrm{d}[k] \cdot 1}_{\text{for reference traj.}})$$

$$\rightsquigarrow A_i = \dots, \quad N_i = 0$$

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TECHNISCHE FAKULTÄT

❶ sample $y_j[k]$ at $t_i = kT + \Delta t_{\mathrm{y},j}[k]$

$$y_{\mathrm{d},j}(t_i^+) = e_j^T y(t_i^-) = e_j^T \left(C_\mathrm{p} x_\mathrm{p}(t_i^-) + \underbrace{w_\mathrm{p}(t_i^-)}_{\text{measurement noise}}\right)$$

$$\underbrace{\begin{bmatrix} x_\mathrm{p}(t_i^+) \\ x_\mathrm{d}(t_i^+) \\ y_\mathrm{d}(t_i^+) \\ u(t_i^+) \\ 1 \end{bmatrix}}_{x(t_i^+)} = \underbrace{\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ e_j e_j^T C_\mathrm{p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{A_i} \underbrace{\begin{bmatrix} x_\mathrm{p}(t_i^-) \\ x_\mathrm{d}(t_i^-) \\ y_\mathrm{d}(t_i^-) \\ u(t_i^-) \\ 1 \end{bmatrix}}_{x(t_i^-)} + N_i^{1/2} v_i,$$

$$N_i = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & e_j e_j^T N_\mathrm{p} e_j e_j^T & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}.$$

1. sample $y_j[k]$ at $t_i = kT + \Delta t_{\mathrm{y},j}[k]$

$$y_{\mathrm{d},j}(t_i^+) = e_j^T y(t_i^-) = e_j^T \left(C_\mathrm{p} x_\mathrm{p}(t_i^-) + \underbrace{w_\mathrm{p}(t_i^-)}_{\text{measurement noise}}\right)$$

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$$A_i = \dots, \quad N_i \neq 0$$

2. update controller state $x_\mathrm{d}[k+1]$ at $t_i = (k + \frac{1}{2})T$

$$x_\mathrm{d}(t_i^+) = x_\mathrm{d}[k+1] = A_\mathrm{d}[k] x_\mathrm{d}(t_i^-) + B_\mathrm{d}[k] y_\mathrm{d}(t_i^-) + \underbrace{f_\mathrm{d}[k] \cdot 1}_{\text{for reference traj.}}$$

$$\rightsquigarrow A_i = \dots, \quad N_i = 0$$

3. output $u_j[k+1]$ at $t_i = (k+1)T + \Delta t_{\mathrm{u},j}[k]$

$$u_{\mathrm{d},j}(t_i^+) = u_j[k+1] = e_j^T (C_\mathrm{d}[k] x_\mathrm{d}(t_i^-) + \underbrace{g_\mathrm{d}[k] \cdot 1}_{\text{for reference traj.}})$$

$$\rightsquigarrow A_i = \dots, \quad N_i = 0$$

- linearized inverse pendulum

$$A_\mathrm{p} = \begin{bmatrix} 0 & 1 \\ \omega_0^2 & -2\xi\omega_0 \end{bmatrix}, \; B_\mathrm{p} = \begin{bmatrix} 0 \\ \omega_0/9.81 \end{bmatrix}, \; G_\mathrm{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_\mathrm{p} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$H = 10^{-3}, \; N_\mathrm{p} = 10^{-6}, \omega_0 = \pi, \xi = 0.5$$

- hold upper equilibrium $(x_\mathrm{r} = 0, u_\mathrm{r} = 0)$
- discrete-time observer-based state feedback
- dangerously fast pole placement:
  - plant: $\{1.94, -5.08\}$
  - controller: $\{-10, -11\}$, observer: $\{-20, -22\}$ (continuous-time equivalents, map via $\lambda_d = \exp(\lambda T)$)
- cost weighting: $Q = 550I, \; R = 0.8 \rightsquigarrow x$ and u "equally expensive"
- stationary cost without delays $\mathcal{E}\{J\} = 1$
- $T = 0.2$

- notation: $\underset{\text{random variables}}{\mathcal{E}}\{\dots\}$
- prior result: expectation w.r.t. noise $\mathcal{N}$

$$\underbrace{\underset{\mathcal{N}}{\mathcal{E}}\{x(t^+)x^T(t^+)\}}_{P(t^+)} = \underset{(0,t]}{F}\Big(\underbrace{\underset{\mathcal{N}}{\mathcal{E}}\{x(0)x^T(0)\}}_{x_0x_0^T}, \mathcal{T}\Big) \quad \text{for } \mathcal{T} \text{ known}$$

- $\mathcal{N}$, $\mathcal{T}$ independent $\rightsquigarrow$ equals conditional expectation

$$\mathcal{E}\{x^T(t^+)x(t^+)|\mathcal{T}\} = \underset{(0,t]}{F}\left(x_0x_0^T, \mathcal{T}\right) \tag{1}$$

- Desired result: unconditional expectation

$$\underset{\mathcal{N},\mathcal{T}}{\mathcal{E}}\{x(t^+)x^T(t^+)\} = \underset{\mathcal{T}}{\mathcal{E}}\left\{\underset{\mathcal{N}}{\mathcal{E}}\{x(t^+)x^T(t^+)|\mathcal{T}\}\right\}$$

$$\overset{(1)}{=} \underset{\mathcal{T}}{\mathcal{E}}\left\{\underset{(0,t]}{F}\left(x_0x_0^T, \mathcal{T}\right)\right\}$$

- notation: $\underset{\text{random variables}}{\mathcal{E}} \{\dots\}$
- prior result: expectation w.r.t. noise $\mathcal{N}$

$$\underbrace{\underset{\mathcal{N}}{\mathcal{E}}\left\{x(t^+)x^T(t^+)\right\}}_{P(t^+)} = \underset{(0,t]}{F}\left(\underbrace{\underset{\mathcal{N}}{\mathcal{E}}\left\{x(0)x^T(0)\right\}}_{x_0 x_0^T}, \mathcal{T}\right) \quad \text{for } \mathcal{T} \text{ known}$$

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$$\mathcal{E}\{x^T(t^+)x(t^+)|\mathcal{T}\} = \underset{(0,t]}{F}\left(x_0 x_0^T, \mathcal{T}\right) \tag{1}$$

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$$\underset{\mathcal{N},\mathcal{T}}{\mathcal{E}}\left\{x(t^+)x^T(t^+)\right\} = \underset{\mathcal{T}}{\mathcal{E}}\left\{\underset{\mathcal{N}}{\mathcal{E}}\left\{x(t^+)x^T(t^+)|\mathcal{T}\right\}\right\}$$

$$\overset{(1)}{=} \underset{\mathcal{T}}{\mathcal{E}}\left\{\underset{(0,t]}{F}\left(x_0 x_0^T, \mathcal{T}\right)\right\}$$

- notation: $\underset{\text{random variables}}{\mathcal{E}} \{\dots\}$
- prior result: expectation w.r.t. noise $\mathcal{N}$

$$\underbrace{\underset{\mathcal{N}}{\mathcal{E}} \{x(t^+)x^T(t^+)\}}_{P(t^+)} = \underset{(0,t]}{F} \Big(\underbrace{\underset{\mathcal{N}}{\mathcal{E}} \{x(0)x^T(0)\}}_{x_0 x_0^T}, \mathcal{T} \Big) \quad \text{for } \mathcal{T} \text{ known}$$

- $\mathcal{N}$, $\mathcal{T}$ independent $\leadsto$ equals conditional expectation

$$\mathcal{E}\{x^T(t^+)x(t^+)|\mathcal{T}\} = \underset{(0,t]}{F} \left(x_0 x_0^T, \mathcal{T}\right) \tag{1}$$

- Desired result: unconditional expectation

$$\underset{\mathcal{N},\mathcal{T}}{\mathcal{E}} \{x(t^+)x^T(t^+)\} = \underset{\mathcal{T}}{\mathcal{E}} \left\{ \underset{\mathcal{N}}{\mathcal{E}} \{x(t^+)x^T(t^+)|\mathcal{T}\} \right\}$$

$$\overset{(1)}{=} \underset{\mathcal{T}}{\mathcal{E}} \left\{ \underset{(0,t]}{F} \left(x_0 x_0^T, \mathcal{T}\right) \right\}$$

properties of $F_{(a,\,b]}$, the covariance evolution for det. timing:

- splitting ($\approx$ semigroup operator):

$$P(b^+) = \underset{(0,\,b]}{F}(x_0 x_0^T, \mathcal{T}) = \underset{(a,\,b]}{F}\left(\underbrace{\underset{(0,\,a]}{F}\left(x_0 x_0^T, \mathcal{T}_{(0,\,a]}\right)}_{P(a^+)}, \mathcal{T}_{(a,\,b]}\right) \tag{2}$$

- "matrix-affine" in start covariance P

$$\underset{(a,\,b]}{F}(P, \mathcal{T}) = M_1^T\, P\, M_1 + M_2 \quad \text{with } M_{1,2} \text{ dep. on } \mathcal{T}, a, b, H, N_\mathrm{p}.$$

$$\cdots \Rightarrow \underset{\mathcal{N}}{\mathcal{E}}\left\{F(P, \mathcal{T})\right\} = F\left(\underset{\mathcal{N}}{\mathcal{E}}\left\{P\right\}, \mathcal{T}\right)$$